

Analytical study of holographic superconductor in Born-Infeld electrodynamics with backreaction

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We generalize the analytical investigation on the properties of s -wave holographic superconductors in the presence of Born-Infeld nonlinear electrodynamics by taking the backreaction into account. We find that in the presence of nonlinear gauge field, one can still employ the analytical method when the backreaction is turned on. Our calculation is based on the Sturm-Liouville eigenvalue problem, which is a variational method. For the system under consideration, we obtain the relation between the critical temperature and the charge density. We find that both backreaction and Born-Infeld parameters decrease the critical temperature of the superconductor and make the condensation harder. At the end of paper, we calculate the critical exponent associated with the condensation near the critical temperature and find that it has the universal value $1/2$ of the mean field theory.

I. INTRODUCTION

The idea of gauge/gravity duality [1] for analyzing the real-world has been one of the remarkable achievements which emerges from development in string theory. It has been well-established that such a duality can provide a novel method for calculating the properties of the superconductors by using a dual classical gravity description. In particular, it can shed some light on the unsolved mysteries in modern condensed matter physics, namely, the mechanism of the high temperature superconductors. The investigation on this subject has got a lot of enthusiasm in the past decades [2–4].

A great step in this direction was put forward by Hartnoll, et. al., [5, 6] who disclosed that some properties of strongly coupled superconductors can be potentially described by classical general relativity living in one higher dimension. This novel idea is usually called *holographic superconductors*. The holographic s -wave superconductor model known as Abelian-Higgs model was first realized in [5, 6]. According to the anti-de Sitter/conformal field theories (AdS/CFT) correspondence, in the gravity side, a Maxwell field and a charged scalar field are introduced to describe the $U(1)$ symmetry and the scalar operator in the dual field theory, respectively. This holographic model undergoes a phase transition from black hole with no hair (normal phase/conductor phase) to the case with scalar hair at low temperatures (superconducting phase). Following [5, 6], an overwhelming number of papers have been appeared which try to apply this novel idea for understanding the strongly coupled holographic superconductors in different setups [7–16]. For a review on the holographic superconductors see [17–19].

On the other hand, there have been a lot of interest in studying the high order correction terms related to the gauge field, in the holographic superconductors. The motivation is to investigate the effects of the nonlinear electrodynamics on the scalar condensation. The effects of Born-Infeld nonlinear electrodynamics on the holographic superconductors has been studied numerically in [20]. Based on the Sturm-Liouville eigenvalue problem, several properties of holographic s -wave superconductors in the background of a Schwarzschild-AdS spacetime and in the presence of Born-Infeld nonlinear electrodynamics were analytically studied in [21]. Similar studies were also done when the nonlinear electrodynamics is in the form of power-Maxwell field [22]. It was shown that the larger power parameter q for the power-Maxwell field makes it harder for the scalar hair to be condensed [22]. Other studies on the holographic superconductors in the presence of nonlinear electrodynamics were carried out in [23, 24].

It is worth noting that most investigations on the holographic superconductors focus on the probe limit in which the scalar and gauge field do not back react on the metric background. Another interesting behavior is when the backreaction of the matter fields on the background geometry is taken into account. This might bring rich physics in the holographic model away from the probe limit. It was shown that when the backreaction is taken into account, even the uncharged scalar field can form a condensate in the $(2+1)$ -dimensional holographic superconductor model [6]. Analytical and numerical investigations, based on the both matching and Sturm-Liouville method, have been carried out for calculating the critical temperature of the holographic superconductor and other physical quantities when the backreaction is turned on [25–31]. Employing the variational method for the Sturm-Liouville eigenvalue problem, the properties of the holographic superconductors with backreaction and with linear Maxwell field were investigated analytically in [32].

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In this paper, we would like to extend the investigations on the holographic superconductors with backreaction by replacing the linear Maxwell field with the nonlinear Born-Infeld electrodynamics. We shall employ the Sturm-Liouville eigenvalue problem to analytically investigate the properties of these holographic superconductors. We find the critical temperature and critical exponent of the holographic superconductor in the presence of Born-Infeld electrodynamics with backreaction. This eventually helps us to consider the strength of both Born-Infeld and backreaction parameters on the condensation of the holographic superconductor.

This paper is structured as follows. In section II, we present action and basic field equations of the holographic superconductors when the Born-Infeld field and scalar field backreact on the metric. In section III, we analytically investigate the properties of these holographic superconductors by using the Sturm-Liouville method and obtain the critical temperature and charge density. In section IV, we calculate the critical exponent and the condensation values of the holographic superconductor. We summarize our results in section V.

II. BASIC EQUATIONS OF HOLOGRAPHIC SUPERCONDUCTORS WITH BACKREACTIONS

Our starting point is the following action in which gravity is coupled to a charged, complex scalar field and Born-Infeld nonlinear electrodynamics,

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2\kappa^2} (R - 2\Lambda) + L_{BI} - |\nabla\psi - iqA\psi|^2 - m^2|\psi|^2 \right], \quad (1)$$

where $\kappa^2 = 8\pi G_4$ is the 4-dimensional gravitational constant, $\Lambda = -3/L^2$ is the cosmological constant and R is the Ricci scalar. Here A and ψ are, respectively, the gauge and scalar field with charge q . The Lagrangian density of the Born-Infeld electrodynamics is defined as

$$L_{BI} = \frac{1}{b} \left(1 - \sqrt{1 + \frac{bF}{2}} \right), \quad (2)$$

where $F = F_{\mu\nu}F^{\mu\nu}$ and $F^{\mu\nu}$ is the electromagnetic field tensor. The constant b is the Born-Infeld coupling parameter which indicates the strength of the nonlinearity. When $b \rightarrow 0$ the Lagrangian of the Born-Infeld reduces to the standard Maxwell Lagrangian, $L_M = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$. Varying the action (1) with respect to the metric, scalar and electrodynamic fields leads to the following field equations

$$\begin{aligned} R^{\mu\nu} - \frac{g^{\mu\nu}}{2}R - \frac{3g^{\mu\nu}}{L^2} &= \frac{\kappa^2}{b}g^{\mu\nu} \left(1 - \sqrt{1 + \frac{bF}{2}} \right) + \frac{\kappa^2}{\sqrt{1 + \frac{bF}{2}}} F_{\sigma}^{\mu} F^{\sigma\nu} \\ -\kappa^2 g^{\mu\nu} m^2 \psi^2 - \kappa^2 g^{\mu\nu} |\nabla\psi - iqA\psi|^2 &+ \kappa^2 [(\nabla^\nu + iqA^\nu)\psi^*(\nabla^\mu - iqA^\mu)\psi + \mu \leftrightarrow \nu], \end{aligned} \quad (3)$$

$$(\nabla_\mu - iqA_\mu)(\nabla^\mu - iqA^\mu)\psi - m^2\psi = 0, \quad (4)$$

$$\nabla_\nu \left(\frac{F^{\nu\mu}}{\sqrt{1 + \frac{bF}{2}}} \right) = iq \left[\psi^*(\nabla^\mu - iqA^\mu)\psi - \psi(\nabla^\mu + iqA^\mu)\psi^* \right]. \quad (5)$$

In the limiting case where $b \rightarrow 0$, the above field equations reduce to the equations of holographic superconductor in Maxwell theory [6]. We take the following metric ansatz for a planar black hole with backreaction [6]

$$ds^2 = -f(r)e^{-\chi(r)}dt^2 + \frac{dr^2}{f(r)} + r^2(dx^2 + dy^2), \quad (6)$$

where $f(r)$ and $\chi(r)$ are functions of r only. The electromagnetic field and the scalar field can be chosen as [5]

$$A_\mu = (\phi(r), 0, 0, 0) \quad \psi = \psi(r). \quad (7)$$

It is worth noting that because of gauge freedom, we can choose ψ to be a real scalar field. The Hawking temperature of the black hole, which will be interpreted as the temperature of the CFT, is given by

$$T = \frac{f'(r_+)e^{-\chi(r_+)/2}}{4\pi}, \quad (8)$$

where the prime denotes the derivative with respect to r , and r_+ is the black hole horizon defined by $f(r_+) = 0$. Inserting the metric (6) as well as (7) in the field equations (3)-(5), the components of the Einstein equations lead to

$$\chi' + 2r\kappa^2 \left(\psi'^2 + \frac{q^2 e^{\chi} \phi^2 \psi^2}{f^2} \right) = 0, \quad (9)$$

$$f' - \left(\frac{3r}{L^2} - \frac{f}{r} \right) + r\kappa^2 \left[m^2 \psi^2 - \frac{1}{b} + \frac{1}{b\sqrt{1 - b e^{\chi} \phi'^2}} + f \left(\psi'^2 + \frac{q^2 e^{\chi} \phi^2 \psi^2}{f^2} \right) \right] = 0, \quad (10)$$

while the scalar and gauge field equations become

$$\phi'' + \frac{2}{r} \phi' (1 - b \phi'^2 e^{\chi}) + \frac{\phi' \chi'}{2} - \frac{2q^2 \psi^2}{f} \phi (1 - b \phi'^2 e^{\chi})^{3/2} = 0, \quad (11)$$

$$\psi'' + \left(\frac{2}{r} - \frac{\chi'}{2} + \frac{f'}{f} \right) \psi' - \frac{m^2}{f} \psi + \frac{q^2 \phi^2 e^{\chi}}{f^2} \psi = 0. \quad (12)$$

One may note that for $b \rightarrow 0$, the field equations (9) -(12) restore the field equations of holographic superconductor with backreaction in Maxwell theory [32], as expected. In this paper, we would like to consider the backreaction of the bulk fields on the background metric that describes a charged Born-Infeld black hole in the AdS bulk. We re-scale the bulk fields ϕ , ψ and the Born-Infeld coupling parameter b as $\phi \rightarrow \phi/q$, $\psi \rightarrow \psi/q$ and $b \rightarrow q^2 b$. Under these transformations, the form of the gauge and the scalar field equations do not change, but the gravitational coupling in the Einstein equation changes $\kappa^2 \rightarrow \kappa^2/q^2$. In general the probe limit is defined as $\kappa^2/q^2 \rightarrow 0$. There are two methods to include the backreaction of matter fields on the metric. The first method is to consider $\kappa^2 = 1$ and choose a finite value of q^2 as described in [6]. In this approach, the probe limit is equivalent to letting $q \rightarrow \infty$ [32]. In the second method, one can fix $q^2 = 1$ [33] and consider finite values of the parameter κ^2 . In this case the probe limit corresponds to letting $\kappa^2 \rightarrow 0$. In this paper, we adopt the second approach to fix the backreaction parameter to be κ^2 . Therefore, in the limiting case where $\kappa^2 \rightarrow 0$, the field equations restore those of the holographic superconductors in Born-Infeld electrodynamics in the probe limit [21].

For the normal phase where $\psi(r) = 0$, from Eq. (9) we find that χ is a constant and the metric becomes the Reissner-Nordström AdS black hole as the Born-Infeld nonlinear parameter b approaches to zero. Thus, we have

$$f(r) = \frac{r^2}{L^2} - \frac{1}{r} \left(\frac{r_+^3}{L^2} + \frac{\kappa^2 \rho^2}{2r_+} \right) + \frac{\kappa^2 \rho^2}{2r^2}, \quad \phi \approx \mu - \frac{\rho}{r}, \quad (13)$$

where μ and ρ are interpreted as the chemical potential and charge density in the holographic superconductor [5]. Note that when the Born-Infeld factor is not equal to zero, the solution is the Born-Infeld-AdS black hole.

Since we are interested in getting solution for superconducting phase where $\psi \neq 0$, we must impose the appropriate boundary conditions. At the black hole horizon, r_+ , we have $f(r_+) = 0$ and the regularity conditions $\phi(r_+) = 0$ for the gauge field [34], imply the boundary conditions

$$\psi(r_+) = \frac{f'(r_+) \psi'(r_+)}{m^2}, \quad (14)$$

and the coefficients in the metric ansatz satisfy

$$\chi'(r_+) = -2\kappa^2 r_+ \left(\psi'^2 + \frac{e^{\chi(r_+)} \phi'(r_+)^2 \psi(r_+)^2}{f'(r_+)^2} \right), \quad (15)$$

$$f'(r_+) = \frac{3r_+}{L^2} - \kappa^2 r_+ \left[m^2 \psi(r_+)^2 - \frac{1}{b} + \frac{1}{b\sqrt{1 - b \phi'^2(r_+) e^{\chi(r_+)}}} \right]. \quad (16)$$

Far from the horizon boundary, at the spatial infinity where $r \rightarrow \infty$, the asymptotic performance of the solutions are

$$\chi \rightarrow 0, \quad f \approx \frac{r^2}{L^2}, \quad \phi \approx \mu - \frac{\rho}{r}, \quad \psi \approx \frac{\psi_-}{r\Delta_-} + \frac{\psi_+}{r\Delta_+}, \quad (17)$$

where

$$\Delta_{\pm} = \frac{3 \pm \sqrt{9 + 4m^2}}{2}. \quad (18)$$

According to the gauge/gravity duality, ψ can be regarded as the source of the dual operator \mathcal{O} , $\psi_- = \langle \mathcal{O}_- \rangle$ and $\psi_+ = \langle \mathcal{O}_+ \rangle$, respectively. Setting $m^2 = -2$ in (18), we have $\Delta_- = 1$ and $\Delta_+ = 2$. Following [5, 6], we can impose the boundary condition in which either ψ_+ or ψ_- vanishes, so that the theory is stable in the asymptotic AdS region. In the following calculation, we will focus on the condition $\psi_+ = 0$. Moreover, we will consider the values of m^2 which must satisfy the Breitenlohner-Freedman (BF) bound $m^2 \geq -9/4$ [35] for the 4-dimensional spacetime. In the remaining part of this paper we will set $L = 1$.

III. ANALYTICAL INVESTIGATION OF THE HOLOGRAPHIC SUPERCONDUCTOR

In this section, we would like to study the $(2+1)$ -holographic superconductor phase transition in the presence of Born-Infeld nonlinear electrodynamics by taking into account the backreaction of the scalar and gauge field on the metric background. We employ the Sturm-Liouville variational method and investigate the relation between the critical temperature of condensation and the charge density near the phase transition point. In particular, we shall examine the effects of the backreaction as well as the Born-Infeld parameters on the critical temperature. In order to solve Eqs. (9)-(12), we rewrite them in terms of a new dimensionless coordinate, $z = r_+/r$. The result is

$$\chi' - 2\kappa^2 \left(z\psi'^2 + \frac{r_+^2}{z^3 f^2} e^{\chi} \phi^2 \psi^2 \right) = 0, \quad (19)$$

$$f' - \frac{f}{z} + \frac{3r_+^2}{L^2 z^3} - \frac{\kappa^2 r_+^2}{z^3} \left[m^2 \psi^2 - \frac{1}{b} + \frac{1}{b} \frac{1}{\sqrt{1 - b e^{\chi} \frac{z^4}{r_+^2} \phi'^2}} + f \left(\frac{z^4}{r_+^2} \psi'^2 + \frac{1}{f^2} e^{\chi} \phi^2 \psi^2 \right) \right] = 0, \quad (20)$$

$$\phi'' + \frac{\phi' \chi'}{2} + \frac{2b e^{\chi} z^3}{r_+^2} \phi'^3 - \frac{2r_+^2 \psi^2}{z^4 f} \phi \left(1 - \frac{b e^{\chi} z^4}{r_+^2} \phi'^2 \right)^{3/2} = 0 \quad (21)$$

$$\psi'' - \left(\frac{\chi'}{2} - \frac{f'}{f} \right) \psi' - \frac{r_+^2}{z^4} \left(\frac{m^2}{f} - \frac{e^{\chi} \phi^2}{f^2} \right) \psi = 0 \quad (22)$$

where the prime now indicates the derivative with respect to z . In the absence of the backreaction, the solution of Eq. (20) is

$$f(z) = r_+^2 \left(\frac{1}{z^2} - z \right), \quad (23)$$

and Eqs. (21) and (22) reduce to their corresponding equations in Ref. [21]. In the vicinity of the critical temperature, T_c , which the stability is confirmed [36], we can select the order parameter as an expansion parameter because it has a small value [32]

$$\epsilon \equiv \langle \mathcal{O}_i \rangle, \quad (24)$$

with $i = +$ or $i = -$. Since we are interested in solutions where $\psi(r)$ is small, therefore from Eqs. (21) and (22) we can expand the scalar field ψ and the gauge field ϕ as [37]

$$\psi = \epsilon \psi_1 + \epsilon^3 \psi_3 + \epsilon^5 \psi_5 + \dots, \quad (25)$$

$$\phi = \phi_0 + \epsilon^2 \phi_2 + \epsilon^4 \phi_4 + \dots, \quad (26)$$

where $\epsilon \ll 1$. The metric functions $f(z)$ and $\chi(z)$ can also be expanded around the Reissner-Nordström AdS spacetime

$$f = f_0 + \epsilon^2 f_2 + \epsilon^4 f_4 + \dots, \quad (27)$$

$$\chi = \epsilon^2 \chi_2 + \epsilon^4 \chi_4 + \dots \quad (28)$$

For the chemical potential μ , we allow it to be expanded as the following series form

$$\mu = \mu_0 + \epsilon^2 \delta\mu_2 + \dots, \quad (29)$$

where $\delta\mu_2 > 0$. Thus, near the phase transition, the order parameter as a function of the chemical potential can be obtained as

$$\epsilon \approx \left(\frac{\mu - \mu_0}{\delta\mu_2} \right)^{1/2}. \quad (30)$$

It is clear that when μ approaches μ_0 , the order parameter ϵ approaches zero. The phase transition occurs at the critical value $\mu_c = \mu_0$. Note that the critical exponent $\beta = 1/2$ is the universal result from the Ginzburg-Landau mean field theory. At the zeroth order, the equation of motion for ϕ reduces to

$$\phi''(z) + \frac{2bz^3}{r_{+c}^2} \phi'(z) = 0. \quad (31)$$

If we set $\phi'(z) = \xi(z)$, we have

$$\xi'(z) + \frac{2bz^3}{r_{+c}^2} \xi(z) = 0. \quad (32)$$

Integrating the above equation in the interval $[0, 1]$, yields

$$\frac{1}{\xi^2(1)} - \frac{1}{\xi^2(0)} = \frac{b}{r_{+c}^2}, \quad (33)$$

where $\xi = \xi(0)$ at $z = 0$ and $\xi = \xi(1)$ at $z = 1$, and from Eq. (13) we have

$$\phi'(0) = \xi(0) \approx -\frac{\rho}{r_+} = -\frac{\rho}{r_{+c}}, \quad (34)$$

at $T = T_c$. Also, from Eqs. (33) and (34), we obtain

$$\frac{1}{\xi^2(1)} = \frac{b}{r_{+c}^2} + \left(\frac{r_{+c}}{\rho} \right)^2. \quad (35)$$

Integrating Eq. (32) in the interval $[1, z]$, after using Eq. (35) we arrive at

$$\xi(z) = \phi'(z) = -\frac{\lambda r_{+c}}{\sqrt{1 + b\lambda^2 z^4}}, \quad (36)$$

where we have taken the negative sign in the expression for $\phi'(z)$ since $\phi'(0)$ is negative at $z = 0$ and

$$\lambda = \frac{\rho}{r_{+c}^2}. \quad (37)$$

Integrating Eq. (36) from $z' = 1$ to $z' = z$, we get

$$\phi(z) = -\int_1^z \frac{\lambda r_{+c}}{\sqrt{1 + b\lambda^2 z'^4}} dz', \quad (38)$$

where we have used the fact that $\phi(z = 1) = 0$. Since the above integral cannot be solved exactly, we shall expand the integrand binomially up to $\mathcal{O}(b)$. We find

$$\phi_0(z) = \lambda r_{+c}(1 - z) \left(1 - \frac{b\lambda^2}{10}(1 + z + z^2 + z^3 + z^4) \right), \quad b\lambda^2 < 1. \quad (39)$$

At the zeroth order, the equation for f has the following solution

$$f_0(z) = r_+^2 g(z) = r_+^2 \left[\frac{1}{z^2} - z - \frac{\kappa^2 \lambda^2}{2} z(1 - z) + \frac{b}{40} \kappa^2 \lambda^4 z(1 - z^5) \right], \quad (40)$$

where we introduce the new function $g(z)$ for simplicity in the following calculations. It is worth noting that we shall assume the deviation from the linear Maxwell field is small. This allows us to keep only the nonlinear parameter b up to the first order. Now, in the first order approximation, the asymptotic AdS boundary conditions ($z \rightarrow 0$) for ψ can be expressed as

$$\psi_1 \approx \frac{\psi_-}{r_+^{\Delta_-}} z^{\Delta_-} + \frac{\psi_+}{r_+^{\Delta_+}} z^{\Delta_+}. \quad (41)$$

In order to match the behavior at the boundary, we can define

$$\psi_1(z) = \frac{\langle \mathcal{O}_i \rangle}{\sqrt{2} r_+^{\Delta_i}} z^{\Delta_i} F(z), \quad (42)$$

where $F(z)$ is a trial function near the boundary $z = 0$ which satisfies the boundary conditions $F(0) = 1$ and $F'(0) = 0$ [38]. Inserting Eq. (42), we can write Eq. (22) as

$$\begin{aligned} F''(z) + \left[\frac{2\Delta}{z} + \frac{g'}{g} \right] F'(z) + \left[\frac{\Delta}{z} \left(\frac{\Delta-1}{z} + \frac{g'}{g} \right) - \frac{m^2}{z^4 g} \right] F(z) \\ + \frac{\lambda^2 (1-z)^2}{z^4 g^2} \left[1 - \frac{b\lambda^2}{5} \left(1 + z + z^2 + z^3 + z^4 \right) \right] F(z) = 0. \end{aligned} \quad (43)$$

This equation can be rewritten as a Sturm-Liouville eigenvalue equation

$$TF'' + T'F' + PF + \lambda^2 QF = 0, \quad (44)$$

where T , P , and Q read

$$T(z) = z^{2\Delta_i+1} \left[2(z^{-3} - 1) - \kappa^2 \lambda^2 (1-z) + \frac{b}{20} \kappa^2 \lambda^4 (1-z^5) \right], \quad (45)$$

$$P(z) = \frac{\Delta_i}{z} \left(\frac{\Delta_i-1}{z} + \frac{g'}{g} \right) - \frac{m^2}{z^4 g}, \quad (46)$$

$$Q(z) = \frac{(1-z)^2}{z^4 g^2} \left[1 - \frac{b\lambda^2}{5} \left(1 + z + z^2 + z^3 + z^4 \right) \right]. \quad (47)$$

According to the boundary conditions for $F(z)$, we can take the trial function as

$$F(z) = 1 - \alpha z^2. \quad (48)$$

The minimal eigenvalue λ^2 is obtained by minimizing the following expression with respect to the coefficient α

$$\lambda^2 = \frac{\int_0^1 T (F'^2 - PF^2) dz}{\int_0^1 T Q F^2 dz}. \quad (49)$$

In order to simplify the following calculation, we will express the backreacting parameter as [24]

$$\kappa_n = n\Delta\kappa, \quad n = 0, 1, 2, \dots \quad (50)$$

where $\Delta\kappa = \kappa_{n+1} - \kappa_n$ is the step size of our iterative procedure. We are interested in finding the effects of the nonlinear corrections on the backreaction term, i.e. we want to obtain the λ^2 up to the order κ^2 ,

$$\kappa^2 \lambda^2 = \kappa_n^2 \lambda^2 = \kappa_n^2 (\lambda^2|_{\kappa_{n-1}}) + \mathcal{O}[(\Delta\kappa)^4]. \quad (51)$$

Here we have set $\kappa_{-1} = 0$ and $\lambda^2|_{\kappa_{-1}} = 0$. We shall perform a perturbative expansion $b\lambda^2$ and retain only the terms that are linear in b such that

$$b\lambda^2 = b(\lambda^2|_{b=0}) + \mathcal{O}(b^2), \quad (52)$$

where $\lambda^2|_{b=0}$ is the value of λ^2 for $b = 0$. In fact, we have only retain the terms that are linear in Born-Infeld parameter b .

According to the definition of T , the critical temperature T_c is given by

$$T_c = \frac{f'(r_{+c})}{4\pi}. \quad (53)$$

Using Eq. (16), we have

$$f'(r_{+c}) = 3r_{+c} - \kappa^2 r_{+c} \left[-\frac{1}{b} + \frac{1}{b\sqrt{1 - b\phi_0'^2(r_{+c})}} \right]. \quad (54)$$

Thus, the critical temperature T_c can be expressed as

$$T_c = \frac{1}{4\pi} \sqrt{\frac{\rho}{\lambda}} \left[3 - \frac{\kappa_n^2(\lambda^2|_{\kappa_{n-1}})}{2} + \frac{1}{8} b \kappa_n^2(\lambda^4|_{\kappa_{n-1}, b=0}) \right], \quad (55)$$

where we have used Eq. (39) as well as relation

$$b\kappa^2\lambda^4 = b\kappa_n^2(\lambda^4|_{\kappa_{n-1}, b=0}) + \mathcal{O}(b^2) + \mathcal{O}[(\Delta\kappa)^4]. \quad (56)$$

In this way we present a complete picture of the critical temperature T_c for the $(2+1)$ -dimensional holographic superconductors in Born-Infeld nonlinear electrodynamics with backreactions. It is important to note that we shall obtain the analytical results by taking the values of $m^2 = -2$, $\Delta_i = \Delta_- = 1$ and setting $\Delta\kappa = 0.05$. Also, the nonlinear parameter is taken as $b = 0, 0.1, 0.2, 0.3$. Obviously, the critical temperature T_c depends on the parameters κ and b . As an example, we bring the details of our calculation for $n = 2$ with different values of b . For $b = 0$, we find

$$\lambda^2 = \frac{1.663870667\alpha^2 - 0.9960856000\alpha + 0.9978253333}{0.7153474994 - 0.1752999949\alpha + 0.03033207562\alpha^2}, \quad (57)$$

whose minimum is 1.2660 at $\alpha = 0.23813$. According to Eq. (55), we get the critical temperature $T_c = 0.2246\sqrt{\rho}$, which is in good agreement with the result of [32]. For $b = 0.1$, we have

$$\lambda^2 = \frac{0.997816 - 0.996076\alpha + 1.66389\alpha^2}{0.688014 - 0.1650222\alpha + 0.0283792\alpha^2}, \quad (58)$$

which attains its minimum 1.314677 at $\alpha = 0.239498$ and the critical temperature reads $0.2225\sqrt{\rho}$. For $b = 0.2$, we arrive at

$$\lambda^2 = \frac{0.997763 - 0.995993\alpha + 1.66384\alpha^2}{0.660686 - 0.154746\alpha + 0.0261691\alpha^2}, \quad (59)$$

whose minimum is 1.36718 at $\alpha = .24091$ and the critical temperature becomes $0.2203\sqrt{\rho}$. For $b = 0.3$, we find

$$\lambda^2 = \frac{0.997455 - 0.995439\alpha + 1.66349\alpha^2}{0.633356 - 0.144469\alpha + 0.239587\alpha^2}, \quad (60)$$

which has a minimum value 1.42378 at $\alpha = 0.242346$, and we can easily get the critical temperature $0.2180\sqrt{\rho}$. Let us summarize our results in table 1. From this table, we see that, for a fixed value of the nonlinear parameter b , the value of the critical temperature decreases with increasing the backreaction parameter κ . Similar behavior between the Born-Infeld parameter b with the critical temperature has also been observed. Namely, for fixed value of the backreaction parameter, the critical temperature decreases with increasing the nonlinear parameter b . Note that we have taken $\kappa_n = n\Delta\kappa$ where $\Delta\kappa = 0.05$. In general, the presence of both Born-Infeld and backreaction decrease the critical temperature and make the condensation harder. The critical temperature T_c obtained here for $\kappa = b = 0$, agrees with the analytical result of Ref. [13] and numerical result in Ref. [5]. Also, considering the effect of b , without the backreaction parameter κ i.e., the probe limit, our results are consistent with those obtained in Ref. [21]. On the other hand in the absence of nonlinear electrodynamics ($b = 0$), our analytical results show a good agreement with the analytical and numerical results obtained in Ref. [32] for the holographic superconductor with backreaction.

| κ_n | $b = 0$ | $b = 0.1$ | $b = 0.2$ | $b = 0.3$ |
|------------|---------|-----------|-----------|-----------|
| 0 | 0.2250 | 0.2228 | 0.2206 | 0.2184 |
| 0.05 | 0.2249 | 0.2227 | 0.2204 | 0.2181 |
| 0.10 | 0.2246 | 0.2225 | 0.2203 | 0.2180 |
| 0.15 | 0.2241 | 0.2220 | 0.2199 | 0.2176 |
| 0.20 | 0.2235 | 0.2214 | 0.2192 | 0.2170 |
| 0.25 | 0.2226 | 0.2208 | 0.2184 | 0.2162 |
| 0.30 | 0.2216 | 0.2196 | 0.2174 | 0.2152 |

Table 1: The critical temperature $T_c/\sqrt{\rho}$ for different values of both parameters b and κ_n .

IV. CRITICAL EXPONENT AND CONDENSATION VALUES

In this section, we would like to compute the condensation values of the condensation operator $\langle \mathcal{O} \rangle$ in the boundary field theory and near the critical temperature. First of all, we write the field equation (21) by using Eq. (42) in the form

$$\phi'' + \frac{2bz^3}{r_+^2} \phi'^3 = \frac{\langle \mathcal{O} \rangle^2}{r_+^2} \mathcal{B}(z) \phi(z), \quad (61)$$

$$\mathcal{B}(z) = \frac{F^2(z)}{1-z^3} \left(1 - \frac{3bz^4}{2r_+^2} \phi'^2(z) \right) \left[1 + \frac{\kappa^2 z^3}{1+z+z^2} \left(\frac{\lambda^2}{2} - \frac{b\lambda^4}{40} \xi(z) \right) \right], \quad (62)$$

where $\xi(z) = 1 + z + z^2 + z^3 + z^4$. Without the backreaction, this equation reduces to the Eq. (42) of Ref.[21]. We shall assume the parameter $\langle \mathcal{O} \rangle^2/r_+^2$ is small. Expanding $\phi(z)$ for the small parameter $\langle \mathcal{O} \rangle^2/r_+^2$, we get

$$\frac{\phi(z)}{r_+} = \lambda(1-z) \left[1 - \frac{b\lambda^2}{10} \xi(z) \right] + \frac{\langle \mathcal{O} \rangle^2}{r_+^2} \chi(z). \quad (63)$$

With the help of Eq. (63), Eq.(61) becomes

$$\begin{aligned} \chi''(z) + 6b\lambda^2 z^3 \chi'(z) &= \frac{\lambda F^2}{1+z+z^2} \left[1 - \frac{b\lambda^2}{10} (\xi(z) + 15z^4) \right] \\ &+ \frac{\lambda F^2 z^3}{(1+z+z^2)^2} \left[\frac{\kappa^2 \lambda^2}{2} - \frac{b\kappa^2 \lambda^4}{40} (3\xi(z) + 30z^4) \right], \end{aligned} \quad (64)$$

with $\chi(1) = 0 = \chi'(1)$. Multiplying this equation by factor $\exp\left(\frac{3b\lambda^2 z^4}{2}\right)$, we arrive at

$$\begin{aligned} \frac{d}{dz} \left(e^{\frac{3b\lambda^2 z^4}{2}} \chi'(z) \right) &= \lambda e^{\frac{3b\lambda^2 z^4}{2}} \frac{F^2}{1+z+z^2} \left[1 - \frac{b\lambda^2}{10} (\xi(z) + 15z^4) \right] \\ &+ \frac{z^3}{1+z+z^2} \left(\frac{\kappa^2 \lambda^2}{2} - \frac{b\kappa^2 \lambda^4}{40} (3\xi(z) + 30z^4) \right). \end{aligned} \quad (65)$$

Integrating both sides of the above equation between $z = 0$ to $z = 1$, yields

$$\chi'(0) = -\lambda \int_0^1 dz e^{\frac{3b\lambda^2 z^4}{2}} \frac{F^2}{1+z+z^2} \left\{ 1 - \frac{b\lambda^2}{10} (\xi(z) + 15z^4) + \frac{z^3}{1+z+z^2} \left[\frac{\kappa^2 \lambda^2}{2} - \frac{b\kappa^2 \lambda^4}{40} (3\xi(z) + 30z^4) \right] \right\}. \quad (66)$$

Combining Eq. (17) with Eq. (63), we have

$$\begin{aligned} \frac{\mu}{r_+} - \frac{\rho}{r_+^2} z &= \lambda(1-z) \left\{ 1 - \frac{b\lambda^2}{10} \xi(z) \right\} + \frac{\langle \mathcal{O} \rangle^2}{r_+^2} \chi(z) \\ &= \lambda(1-z) \left\{ 1 - \frac{b\lambda^2}{10} \xi(z) \right\} + \frac{\langle \mathcal{O} \rangle^2}{r_+^2} (\chi(0) + z\chi'(0) + \dots). \end{aligned} \quad (67)$$

Comparing the coefficient of z on both sides of the Eq. (67), we get

$$\frac{\rho}{r_+^2} = \lambda - \frac{\langle \mathcal{O} \rangle^2}{r_+^2} \chi'(0). \quad (68)$$

Substituting $\chi'(0)$ in the above equation, we reach

$$\frac{\rho}{r_+^2} = \lambda \left\{ 1 + \frac{\langle \mathcal{O} \rangle^2}{r_+^2} \mathcal{A} \right\}, \quad (69)$$

where

$$\mathcal{A} = \int_0^1 dz e^{\frac{3b\lambda^2 z^4}{2}} \frac{F^2}{1+z+z^2} \left\{ 1 - \frac{b\lambda^2}{10} (\xi(z) + 15z^4) + \frac{z^3}{1+z+z^2} \left[\frac{\kappa^2 \lambda^2}{2} - \frac{b\kappa^2 \lambda^4}{40} (3\xi(z) + 30z^4) \right] \right\}. \quad (70)$$

Next, we should compute an expression for r_+ . Considering the fact that T is very close to T_c and using Eqs. (8), (16) and (39), we have

$$r_+ = \frac{4\pi T}{\left[3 - \frac{\kappa^2 \lambda^2}{2} + \frac{b}{8} \kappa^2 \lambda^4 \right]}. \quad (71)$$

Finally, with the help of Eqs. (37), (55), and (71), we get the following expression,

$$T_c^2 - T^2 = \langle \mathcal{O} \rangle^2 \frac{\mathcal{A}}{(4\pi)^2} \left[3 - \frac{\kappa^2 \lambda^2}{2} + \frac{b}{8} \kappa^2 \lambda^4 \right]^2. \quad (72)$$

Therefore, we find

$$\langle \mathcal{O} \rangle = \gamma T_c \sqrt{1 - \frac{T}{T_c}}, \quad (73)$$

where

$$\gamma = \frac{4\pi\sqrt{2}}{\sqrt{\mathcal{A}}} \left[3 - \frac{\kappa^2 \lambda^2}{2} + \frac{b}{8} \kappa^2 \lambda^4 \right]^{-1}. \quad (74)$$

Thus, near the critical point, the condensation operator $\langle \mathcal{O} \rangle$ will satisfy

$$\langle \mathcal{O} \rangle \sim \sqrt{1 - \frac{T}{T_c}}, \quad (75)$$

which holds for various values of both Born-Infeld and backreaction parameters. Also, the critical exponent is identical to the mean field value $1/2$, which implies that the existence of the mentioned parameters do not have any consequence on the second-order phase transition. Considering only the Born-Infeld parameter, the values of \mathcal{A} is in good agreement with the one in Ref. [21]. We summarize the our results in table 2. We see that the condensation value increases as the Born-Infeld parameter b increases for the fixed parameter κ . On the other hand, for fixed value of b the condensation parameter increases with increasing the backreaction parameter. Thus, in both cases the condensation value increases, which shows that the higher Born-Infeld electrodynamics and gravitational backreaction corrections make the condensation to be harder.

| κ_n | $b = 0$ | $b = 0.1$ | $b = 0.2$ | $b = 0.3$ |
|------------|---------|-----------|-----------|-----------|
| 0.0 | 8.07 | 8.18 | 8.31 | 8.47 |
| 0.05 | 8.09 | 8.187 | 8.321 | 8.489 |
| 0.10 | 8.11 | 8.19 | 8.324 | 8.4893 |
| 0.15 | 8.115 | 8.21 | 8.34 | 8.50 |
| 0.20 | 8.13 | 8.24 | 8.37 | 8.53 |
| 0.25 | 8.16 | 8.29 | 8.39 | 8.56 |
| 0.30 | 8.20 | 8.31 | 8.44 | 8.60 |

Table 2: The values of the condensation parameter γ for different values of b and κ_n .

V. CONCLUSIONS

Employing the Sturm-Liouville eigenvalue problem, we have analytically investigated the effects of the Born-Infeld nonlinear gauge field on the properties of $(2+1)$ -dimensional holographic superconductors in the background of AdS black holes. We performed our analysis away from the probe limit, where the scalar and gauge fields back react on the background metric. First, we presented a detailed analysis of solving the coupled equations of motion for the scalar and gauge fields. We obtained the relationship between the critical temperature and the charge density. We performed our calculations up to the first order in the Born-Infeld coupling parameter and up to order κ^2 in the backreaction parameter. We observed that both backreaction and Born-Infeld parameters make the critical temperature of the holographic superconductor smaller. This implies that the condensation formation is affected by both the backreaction and the Born-Infeld coupling parameters. That is to say, the condensation becomes harder in the presence of the backreaction and Born-Infeld parameters. We also found that the critical exponent of the condensation is $1/2$ which is the universal value in the mean field theory. The results obtained in this paper consist with the previous numerical and analytical results in the limiting cases where either the backreaction or the nonlinear parameters are turned off. We expect our analytical results to be confirmed numerically in the near future investigations. It is also interesting to extend this investigation to other type of nonlinear electrodynamics such as exponential, logarithmic and power-Maxwell Lagrangian. These issues are now under investigations and the results will be appeared soon.

Acknowledgments

We thank Shiraz University Research Council. This work has been supported financially by Research Institute for Astronomy and Astrophysics of Maragha (RIAAM), Iran.

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